Fibrational Initial Algebra-Final Coalgebra Coincidence over Initial Algebras: Turning Verification Witnesses Upside Down

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------ **Abstract** ------

The coincidence between initial algebras (IAs) and final coalgebras (FCs) is a phenomenon that underpins various important results in theoretical computer science. In this paper, we identify a general fibrational condition for the IA-FC coincidence, namely in the fiber over an initial algebra in the base category. Identifying (co)algebras in a fiber as (co)inductive predicates, our fibrational IA-FC coincidence allows one to use *coinductive* witnesses (such as invariants) for verifying *inductive* properties (such as liveness). Our general fibrational theory features the technical condition of stability of chain colimits; we extend the framework to the presence of a monadic effect, too, restricting to fibrations of complete lattice-valued predicates. Practical benefits of our categorical theory are exemplified by new "upside-down" witness notions for three verification problems: probabilistic liveness, and acceptance and model-checking with respect to bottom-up tree automata.

1 Introduction

Categorical Algebras and Coalgebras. Categorical algebras and coalgebras are omnipresent in theoretical computer science. For a category $\mathbb{C}$ and an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$, an F-algebra is a $\mathbb{C}$-morphism $a\colon FX \to X$, while an F-coalgebra is $c\colon X \to FX$. These structures occur in many different settings with different $\mathbb{C}$ and F; the identification of such (co)algebras has yielded a number of concrete benefits, such as rigorous system/program semantics, verification methods, and programming language constructs.

One principal use of categorical (co)algebras is as models of *data structures* such as terms and state-based systems. Examples include modeling of inductive datatypes by initial algebras [6], and the theory of coalgebras [18, 26] that captures state-based behaviors. Here, the base category $\mathbb{C}$ is typically that of (structured) sets and (structure-preserving) maps. (In this paper, such a category will constitute a *base category* of a fibration).

Another principal use of (co)algebras is as logical *recursive specifications*. Here the base category $\mathbb{C}$ is typically a complete lattice of truth values (such as $\mathbf{2} = \{\bot, \top\}$) and the functor $F : \mathbb{C} \to \mathbb{C}$ is identified with a monotone function. Liveness properties are modeled by least fixed points (lfp's); safety properties are greatest ones (gfp's); and by the classic Knaster–Tarski theorem, these are identified with initial algebras and final coalgebras, respectively. (In this paper, such a category will appear as a *fibre category* of a fibration).

Initial Algebras and Final Coalgebras. In the above variety of occurrences of (co)algebras, *initial algebras* and *final coalgebras* play key roles. Their definition is by suitable universality: $\beta : F(\mu F) \to \mu F$ is initial if there is a unique algebra morphism from β to an arbitrary algebra $a : FX \to X$; and dually for final coalgebras.

$$
\begin{array}{cc}
\begin{array}{ccc}
F(\mu F) & \dashrightarrow & FX \\
{\scriptstyle \text{init.}}\downarrow{\scriptstyle\cong} & & \downarrow \\
\mu F & \dashrightarrow & X
\end{array}
&
\begin{array}{ccc}
FX & \dashrightarrow & F(\nu F) \\
\uparrow & & {\scriptstyle\cong}\uparrow{\scriptstyle\text{final}} \\
X & \dashrightarrow & \nu F
\end{array}
\end{array}
$$

Their (co)algebra structures are isomorphisms by the Lambek lemma. The latter extends the Knaster–Tarski theorem from lattices to categories.

In many occurrences of (co)algebras in computer science, initial algebras represent *finitary* entities while final coalgebras represent *infinitary* entities. For example, when $\mathbb{C} = \mathbf{Set}$ (the category of sets and functions) and F is a functor that models a datatype constructor, the carrier μF of an initial algebra represents the *inductive datatype* – collecting all finite trees "of shape F" – while the carrier νF of a final coalgebra is for the *coinductive datatype* and collects all (finite and infinite) trees. This intuition is found also in the logical (co)algebras: liveness properties (initial algebras) can be witnessed within finitely many steps, while safety properties (final coalgebras) are verified only after infinitely many steps.

Initial Algebra-Final Coalgebra Coincidence. In this paper, we are interested in the coincidence of an initial algebra and a final coalgebra (the *IA-FC coincidence*). While it may sound unlikely in view of the contrast between finitary and infinitary, the coincidence has been found in different areas in computer science, underpinning fundamental results.

One example is in *domain theory*: cpo-enrichment yields the IA-FC coincidence, which is used to solve recursive domain equations of mixed variance [11, 13, 27, 32]. Another example is in *process semantics*: specifically, in the coalgebraic characterization of finite trace semantics [15], the IA-FC coincidence in some Kleisli categories $\mathbf{Kl}(T)$ has been observed.

Contribution: the Fibrational IF/I Coincidence and Application to Verification Witnesses. In this paper, we identify a general *fibrational* condition for the IA-FC coincidence: under mild assumptions, we have the IA-FC coincidence in the fiber over an initial algebra in the base category (the *IF/I coincidence*). Identifying the base IA as a datatype, and the fibre IA/FC as lfp/gfp specifications, the IF/I coincidence implies the coincidence between *induction* and *coinduction* as reasoning principles, assuming they are over a (finitary) algebraic datatype.

This coincidence allows us to *turn witness notions upside down*, that is, to use coinductive witness notions for establishing inductive properties. In general, inductive witness notions for lfp properties (such as ranking functions) tend to be more complex than coinductive witness notions for gfp properties (such as invariants). When we have the IF/I coincidence, the latter can now be used for lfp properties.

Our technical contributions are as follows. We work with a *fibration* $p : \mathbb{E} \to \mathbb{B}$, where $\mathbb{B}$ is intuitively a category of sets and functions, and $\mathbb{E}$ equips these sets with predicates.

- We identify a general fibrational framework for what we call the *IF/I coincidence* – the coincidence of IAs (lfp predicates) and FCs (gfp predicates) in the fiber over an initial algebra in $\mathbb{B}$ (an inductive datatype). The IF/I coincidence relies only on mild fibrational assumptions, notable among which are *fibredness* of functors and *stability* of certain colimits. Although we restrict fibrations to posetal ones in the main text (§4), a similar result for general fibrations can be shown (Appendix A).

- As a notable class of examples, in §5 we show that the fibration of Ω-valued predicates exhibits the IF/I coincidence (where Ω is an arbitrary complete lattice for truth values). Furthermore, we study the IF/I coincidence in the presence of monadic effects [23], building on the fibrational framework from [4].

- These theoretical results are used to obtain coinductive (invariant-like) witness notions for inductive (lfp, liveness) properties. Specifically, we present new witness notions for probabilistic verification (§6) and verification with tree automata as specifications (§7).

Related Work.   Many works are discussed in the technical sections; we discuss some others.

The work [25] shows uniqueness of fixed points above what is called a *minimal invariant*; the latter corresponds to the lifting of a morphism which is both an initial algebra and a final coalgebra. Our IF/I coincidence can yield such lifting under some assumptions (see Thm. 3.6). The proof in [25] relies on homset enrichment, unlike our fibrational framework.

One of our main ideas is to use the IA-FC coincidence for novel proof methods for recursive specifications (§6–7), mixing lfp's and gfp's. This is pursued also in [8, 31] where *corecursive algebras* induce the lfp-gfp coincidence.

Organization.   After recalling fibrations and the chain construction of initial algebras in §2, we formulate our IF/I coincidence in §3 and present sufficient conditions for the coincidence in §4. In §5, these results are specialized to fibrations of Ω-valued predicates, where we additionally include monadic effects. This paves the way to the concrete applications in §6–7, where we present seemingly new verification techniques for probabilistic liveness and witnesses of tree automata. We defer many proofs to the appendix.

2 Preliminaries

2.1 Fibrations

A fibration $p : \mathbb{E} \to \mathbb{B}$ is a functor that models indexing and substitution. That is, a functor $p : \mathbb{E} \to \mathbb{B}$ can be seen as a family of categories $(\mathbb{E}_X)_{X \in \mathbb{B}}$ that is equipped with substitution functors that change the index X.

In our examples, the base category $\mathbb{B}$ is that of sets and (potentially effectful) functions; and the total category $\mathbb{E}$ models "predicates" over sets in $\mathbb{B}$. We review a minimal set of definitions and results on fibrations. See [19] for details.

▶ **Definition 2.1** (fibre, fibration)**.** Let $p : \mathbb{E} \to \mathbb{B}$ be a functor.

For each $X \in \mathbb{B}$, the *fibre category* (or simply *fibre*) $\mathbb{E}_X$ over X is the category with objects $P \in \mathbb{E}$ such that $pP = X$ and morphisms $f : P \to Q$ such that $pf = \mathrm{id}_X$. An object $P \in \mathbb{E}_X$ is said to be *above* X and a morphism $f \in \mathbb{E}_X$ is said to be *vertical*.

A morphism $f : P \to Q$ in $\mathbb{E}$ is *cartesian* if it satisfies the following universality: for each $g : R \to Q$ in $\mathbb{E}$ and $k : pR \to pP$ in $\mathbb{B}$ with $pg = pf \circ k$, there exists a unique morphism $h : R \to P$ satisfying $g = f \circ h$ and $ph = k$. See the diagram below.

$$
\begin{array}{ll}
\mathbb{E} \quad R \xrightarrow{\ g\ } & \\
p \Big\downarrow \quad {}_{h}\nearrow P \xrightarrow[f]{} Q & \quad l^*Q \xrightarrow{\ \bar{l}\ } Q \\
\mathbb{B} \quad pR \xrightarrow{\ pg\ } & \\
\quad {}_{k}\nearrow pP \xrightarrow[pf]{} pQ & \quad X \xrightarrow[l]{} pQ
\end{array}
$$

The functor $p : \mathbb{E} \to \mathbb{B}$ is a *fibration* if, for each $Q \in \mathbb{E}$ and each $l : X \to pQ$ in $\mathbb{B}$, there exists $l^*Q \in \mathbb{E}$ and a morphism $\bar{l} : l^*Q \to Q$ such that $p\bar{l} = l$ and $\bar{l}$ is cartesian.

The functor $p : \mathbb{E} \to \mathbb{B}$ is an *opfibration* if $p^{\mathrm{op}} : \mathbb{E}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is a fibration. A functor that is both a fibration and an opfibration is called a *bifibration*.

When p is a fibration, the correspondence from Q to l^*Q described above induces the *substitution functor* $l^* : \mathbb{E}_Y \to \mathbb{E}_X$ which replaces the index. The following characterization of bifibrations is useful for us: a fibration p is a bifibration if and only if each substitution functor $l^* : \mathbb{E}_Y \to \mathbb{E}_X$ (often called a *pullback*) has a left adjoint $l_* : \mathbb{E}_X \to \mathbb{E}_Y$ (often called a *pushforward*).

We are interested in reasoning over algebraic datatypes, that is in categorical terms, predicates in $\mathbb{E}_{\mu F}$ over the carrier μF of the initial algebra for $F : \mathbb{B} \to \mathbb{B}$. For this purpose we often consider a tuple $(p, F, \dot{F})$ in the following definition.

▶ **Definition 2.2** ((fibred) lifting)**.** Let $p : \mathbb{E} \to \mathbb{B}$ be a functor and F be an endofunctor on $\mathbb{B}$. We say that an endofunctor $\dot{F}$ on $\mathbb{E}$ is a *lifting* of F along p if $p \circ \dot{F} = F \circ p$ (see above).

Assuming that p is a fibration, a lifting $\dot{F}$ is *fibred* if $\dot{F}$ preserves cartesian morphisms.

In this paper, we focus on a certain class of posetal fibrations called $\mathbf{CLat}_\wedge$-fibrations. They can be seen as *topological functors* [17] whose fibres are posets. This class abstracts treatment of spacial and logical structures.

▶ **Definition 2.3** ($\mathbf{CLat}_\wedge$-fibration)**.** A $\mathbf{CLat}_\wedge$-*fibration* is a fibration $p : \mathbb{E} \to \mathbb{B}$ where each fibre $\mathbb{E}_X$ is a complete lattice and each substitution $f^* : \mathbb{E}_Y \to \mathbb{E}_X$ preserves all meets $\bigwedge$.

In each fibre $\mathbb{E}_X$, the order is denoted by $\leq_X$ or $\leq$. Its least and greatest elements are denoted by $\perp_X$ and $\top_X$; its join and meet are denoted by $\bigvee$ and $\bigwedge$.

The above simple axioms of $\mathbf{CLat}_\wedge$-fibrations induce many useful structures [20, 28]. One of them is that a $\mathbf{CLat}_\wedge$-fibration is always a bifibration whose pushforwards f_* arise essentially by Freyd's adjoint functor theorem. Another one is that $\mathbf{CLat}_\wedge$-fibrations lift colimits. This is proved by [19, Prop. 9.2.2 and Exercise 9.2.4].

▶ **Proposition 2.4.** *Let* $p : \mathbb{E} \to \mathbb{B}$ *be a* $\mathbf{CLat}_\wedge$-*fibration.*
1. *p is a bifibration.*
2. *If $\mathbb{B}$ is (co)complete then $\mathbb{E}$ is also (co)complete and p strictly preserves (co)limits.*

▶ **Example 2.5** ($\mathbf{CLat}_\wedge$-fibration)**.**
- ($\mathbf{Pre} \to \mathbf{Set}$, $\mathbf{Pred} \to \mathbf{Set}$) These forgetful functors are $\mathbf{CLat}_\wedge$-fibrations. Here $\mathbf{Pre}$ is the category of preordered sets $(X, \leq_X)$ and order-preserving functions between them. $\mathbf{Pred}$ is that of predicates: objects are $P \subseteq X$, and morphisms $f : (P \subseteq X) \to (Q \subseteq Y)$ are functions $f : X \to Y$ satisfying $f(P) \subseteq Q$.
- ($\mathbf{ERel} \to \mathbf{Set}$) The functor $\mathbf{ERel} \to \mathbf{Set}$ defined by the change-of-base [19], as shown below, is a $\mathbf{CLat}_\wedge$-fibration. Concretely, $\mathbf{ERel}$ is the category of sets with binary relations $(X, R \subseteq X \times X)$ as objects, and relation-preserving maps as morphisms.

$$
\begin{array}{ccc}
\mathbf{ERel} & \to & \mathbf{Pred} \\
\downarrow & & \downarrow \\
\mathbf{Set} & \xrightarrow{(-)^2} & \mathbf{Set}
\end{array}
$$

 (Domain fibration $d^\Omega : \mathbf{Set}/\Omega \to \mathbf{Set}$) For each complete lattice Ω, we introduce a $\mathbf{CLat}_\wedge$-fibration $d^\Omega : \mathbf{Set}/\Omega \to \mathbf{Set}$ defined as follows.

$$
\begin{array}{ccc}
X & \xrightarrow{\ h\ } & Y \\
 & \underset{\leq_X}{} & \\
f \searrow & & \swarrow g \\
 & \Omega &
\end{array}
$$

Here, $\mathbf{Set}/\Omega$ is a lax slice category defined as follows: objects of $\mathbf{Set}/\Omega$ are pairs $(X, f : X \to \Omega)$ of a set and a function (an "Ω-valued predicate on X"); we shall often write simply $f : X \to \Omega$ for the pair (X, f). Its morphisms from $f : X \to \Omega$ to $g : Y \to \Omega$ are functions $h : X \to Y$ such that $f \leq_X g \circ h$, as shown above.

Then d_Ω is the evident forgetful functor, extracting the upper part of the above triangle. The order $\leq_X$ used there is the pointwise order between functions of the type $X \to \Omega$; the same order $\leq_X$ defines the order in each fiber $(\mathbf{Set}/\Omega)_X = \mathbf{Set}(X, \Omega)$. Following [4, Def. 4.1], we call d^Ω a *domain fibration* (from the lax slice category).

2.2 Chain Construction of Initial Algebras

▶ **Definition 2.6** (chain-cocomplete category). A category $\mathbb{C}$ is *chain-cocomplete* if $\mathbb{C}$ has a colimit of every chain. We write 0 for a colimit of the empty chain (i.e. an initial object).

Noteworthy is that chain-cocompleteness is equivalent to existence of an initial object and filtered colimits, see [2, Cor. 1.7] for further details.

▶ **Definition 2.7** (initial chain [1], [3, Def. 3.2]). Let $\mathbb{C}$ be a chain-cocomplete category, and $F : \mathbb{C} \to \mathbb{C}$ be an endofunctor. The *initial chain* of F is the following diagram:

$$
0 \xrightarrow{\alpha_{0,1}} F0 \xrightarrow{\alpha_{1,2}} \cdots \longrightarrow F^\lambda 0 \xrightarrow{\alpha_{\lambda,\lambda+1}} \cdots. \tag{1}
$$

This consists of the following.
 (Objects) It has objects $F^i 0$ for each $i \in \mathrm{Ord}$ (where Ord is the category of ordinals), defined by $F^0 0 = 0$, $F^{i+1} 0 = F(F^i 0)$, and for a limit ordinal i, $F^i 0 = \mathrm{colim}_{j<i} F^j 0$.
 (Morphisms) It has morphisms $\alpha_{i,j} : F^i 0 \to F^j 0$ for all ordinals i, j such that $i \leq j$, defined inductively on i. (Base case) $\alpha_{0,j} : 0 \to F^j 0$ is the unique morphism. (Step case) $\alpha_{i+1,j+1}$ is $F\alpha_{i,j}$; for a limit ordinal j, $\alpha_{i+1,j}$ is from the colimiting cocone for $F^j 0$. (Limit case) When i is a limit ordinal, $\alpha_{i,j}$ is induced by universality of $F^i 0 = \mathrm{colim}_{k<i} F^k 0$.

If $\alpha_{\lambda,\lambda+1}$ is an isomorphism, then we say that the initial chain of F *converges* in λ steps.

▶ **Proposition 2.8** (from [1], [3, Thm. 3.5]). *In the setting of Def. 2.7, assume that the initial chain converges in λ steps. Then $\alpha_{\lambda,\lambda+1}^{-1} : F^{\lambda+1}0 \overset{\cong}{\Rightarrow} F^\lambda 0$ is an initial F-algebra.*

The dual of the initial chain in Def. 2.7 is called *the final chain*. This also satisfies the dual of Prop. 2.8 (yielding final coalgebras), see [3, Def. 3.20 and Thm. 3.21].

The converse of Prop. 2.8 holds if we restrict to $\mathbf{Set}$.

▶ **Proposition 2.9** (from [30], [3, Cor. 3.16]). *A set functor has an initial algebra if and only if the initial chain converges.*

We often write μF for the carrier of an initial algebra of F.

The next basic lemma is important for us. Its dual (for coalgebras) is observed e.g. in [14].

▶ **Lemma 2.10.** *Assume that $p : \mathbb{E} \to \mathbb{B}$ is a fibration, that both $\mathbb{E}$ and $\mathbb{B}$ are chain-cocomplete, and that p strictly preserves chain colimits. Let $\dot{F}$ be a lifting of $F : \mathbb{B} \to \mathbb{B}$ along p.*
   Consider the following initial chains.

$$
\begin{array}{cccccc}
\mathbb{E} & 0 \xrightarrow{\dot{\alpha}_{0,1}} \dot{F}0 \xrightarrow{\dot{\alpha}_{1,2}} \cdots \longrightarrow \dot{F}^\lambda 0 \xrightarrow{\dot{\alpha}_{\lambda,\lambda+1}} \cdots \\
p\downarrow & \\
\mathbb{B} & 0 \xrightarrow{\alpha_{0,1}} F0 \xrightarrow{\alpha_{1,2}} \cdots \longrightarrow F^\lambda 0 \xrightarrow{\alpha_{\lambda,\lambda+1}} \cdots
\end{array}
$$

1. *We have $\alpha_{i,j} = p\dot{\alpha}_{i,j}$ for all ordinals i, j with $i < j$.*
2. *Moreover, if the upper initial chain for $\dot{F}$ converges and yield an initial $\dot{F}$-algebra $\dot{\alpha} : \dot{F}(\mu\dot{F}) \to \mu\dot{F}$, then $p\dot{\alpha} : Fp(\mu\dot{F}) \to p(\mu\dot{F})$ is an initial F-algebra.*

3 Initial Algebra-Final Coalgebra Coincidence over Initial Algebras

In this section, we formulate our target coincidence called the *IF/I coincidence*. It is a fibrational IA-FC coincidence over an initial algebra.

▶ **Definition 3.1** (IF/I coincidence). Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, and $\dot{F}$ be a lifting of F. We say that the tuple $\left(p : \mathbb{E} \to \mathbb{B}, \ F : \mathbb{B} \to \mathbb{B}, \ \dot{F} : \mathbb{E} \to \mathbb{E} \right)$ satisfies the *IA-FC coincidence over an initial algebra (IF/I coincidence, for short)* if the following is satisfied.
1. There is an initial F-algebra $\beta : F(\mu F) \overset{\cong}{\Rightarrow} \mu F$.
2. There is an initial $\dot{F}$-algebra $\dot{\beta} : \dot{F}(\mu\dot{F}) \overset{\cong}{\Rightarrow} \mu\dot{F}$ above β.
3. Moreover, $\dot{\beta}^{-1}$ is final over β^{-1} in the following sense: for each $\dot{F}$-coalgebra γ above β^{-1} (shown below diagram on the left), there exists a unique vertical coalgebra morphism f from γ to $\dot{\beta}^{-1}$ (below diagram on the right, where vertical means $pf = \mathrm{id}_{\mu F}$).

$$
\dot{F} \circlearrowleft \mathbb{E}
\qquad\qquad\qquad\qquad
\dot{F}(\mu\dot{F}) \overset{\dot{\beta}^{-1}}{\underset{\cong}{\Leftarrow}} \mu\dot{F}
$$

$$
p\downarrow \qquad \dot{F}P \xleftarrow{\gamma} P \qquad \Longrightarrow \qquad \dot{F}f\uparrow \qquad \uparrow f
$$

$$
F \circlearrowleft \mathbb{B} \quad F(\mu F) \overset{\beta^{-1}}{\underset{\cong}{\Leftarrow}} \mu F; \qquad\qquad F(\mu F) \overset{\beta^{-1}}{\underset{\cong}{\Leftarrow}} \mu F;
$$

IF/I Coincidence in Fibrations of (Co)algebras.   The IF/I coincidence in Def. 3.1 is nicely organized in terms of *fibrations of (co)algebras*: the last two conditions in Def. 3.1 can be stated succinctly in advanced fibrational terms.
   Given a functor $F : \mathbb{B} \to \mathbb{B}$, $\mathbf{Alg}(F)$ is the category of *F-algebras*, where an object is a pair $(X \in \mathbb{B}, a : FX \to X)$ and a morphism from (X, a) to (Y, b) is $f : X \to Y$ such that $b \circ Ff = f \circ a$. Dually, $\mathbf{Coalg}(F)$ is the category of *F-coalgebras*, where an object is $(X \in \mathbb{B}, c : X \to FX)$ and a morphism from (X, c) to (Y, d) is f such that $d \circ f = Ff \circ c$.
   Then a fibration p and a fibred lifting $\dot{F}$ yield fibrations of (co)algebras.

▶ **Proposition 3.2** (from [14, Prop. 4.1]). *A lifting $\dot{F}$ of F along a fibration p induces functors* $\mathbf{Alg}(p) : \mathbf{Alg}(\dot{F}) \to \mathbf{Alg}(F)$ *and* $\mathbf{Coalg}(p) : \mathbf{Coalg}(\dot{F}) \to \mathbf{Coalg}(F)$, *given by*

$$
\mathbf{Alg}(p) : (\dot{F}P \xrightarrow{q} P) \longmapsto (FpP = p\dot{F}P \xrightarrow{pq} pP),
$$

$$
\mathbf{Coalg}(p) : (P \xrightarrow{r} \dot{F}P) \longmapsto (pP \xrightarrow{pr} p\dot{F}P = FpP).
$$

The functor $\mathbf{Alg}(p)$ is a fibration. If additionally $\dot{F}$ is a fibred lifting, then $\mathbf{Coalg}(p)$ is a fibration, too. For an opfibration p, we have a result dual to the above: $\mathbf{Coalg}(p)$ is an opfibration; so is $\mathbf{Alg}(p)$ if $\dot{F}$ is an opfibred lifting (preserving cocartesian morphisms).

The functor $\mathbf{Coalg}(p)\colon \mathbf{Coalg}(\dot{F}) \to \mathbf{Coalg}(F)$ in Prop. 3.2 plays an important role in the following development. It is thought of as a functor where

- (following the coalgebraic tradition) state-based transition systems $c\colon X \to FX$ and behavior-preserving morphisms between them populate the base category $\mathbf{Coalg}(F)$, and
- *invariants* – i.e. predicates $P \in \mathbb{E}_X$ over X that are *preserved* by transitions – populate the total category $\mathbf{Coalg}(\dot{F})$. The arrows in $\mathbf{Coalg}(\dot{F})$ are logical implication of invariants.

The following reformulation is proved in [21, Appendix C.2], together with technical remarks.

▶ **Proposition 3.3.** *The following is equivalent to Cond. 2 and 3 in Def. 3.1, respectively.*
2'. *There is an initial object $\dot{\beta}$ in the fiber $\mathbf{Alg}(\dot{F})_\beta$.*
3'. *$\dot{\beta}^{-1}$ is a final object in the fiber $\mathbf{Coalg}(\dot{F})_{\beta^{-1}}$.*

IF/I Coincidence in a $\mathbf{CLat}_\wedge$-fibration. Here we shall rewrite conditions in Def. 3.1 for $\mathbf{CLat}_\wedge$-fibrations. But first, we need the following investigation of these conditions.

An initial $\dot{F}$-algebra lying above an initial F-algebra is a norm (Cond. 1–2; cf. Lem. 2.10). What is special is the finality of the initial $\dot{F}$-algebra (Cond. 3). The intuition of the latter is the following:

an lfp and a gfp coincide, in the fiber over the base initial algebra.

Intuitively, P with $(\gamma\colon P \to \dot{F}P) \in \mathbf{Coalg}(\dot{F})_{\beta^{-1}}$ is an *invariant* – it is a predicate that is preserved by the transition β^{-1}. Indeed, the morphism γ is equivalent to a morphism

$\gamma^\dagger\colon P \longrightarrow (\beta^{-1})^*\dot{F}P$ over $\mathrm{id}_{\mu F}$, that is,

$P \leq (\beta^{-1})^*\dot{F}P$ if the fibration $p\colon \mathbb{E} \to \mathbb{B}$ is posetal,

by pulling back along β^{-1}. The latter inequality signifies that P is an invariant. This equivalence is formulated as follows.

▶ **Lemma 3.4.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration and $\dot{F}$ be a lifting of F along p. For any isomorphism $\alpha : X \overset{\cong}{\Rightarrow} FX$ in $\mathbb{B}$, $\mathbf{Alg}(\dot{F})_{\alpha^{-1}} \cong \mathbf{Alg}(\alpha^*\dot{F})$ and $\mathbf{Coalg}(\dot{F})_\alpha \cong \mathbf{Coalg}(\alpha^*\dot{F})$.*

Therefore, Cond. 3 requires that $\dot{\beta}^{-1}$ gives a greatest invariant. In view of the Knaster–Tarski theorem (that a greatest post-fixed point is a greatest *fixed* point), this means that $\dot{\beta}^{-1}$ is a gfp if p is a $\mathbf{CLat}_\wedge$-fibration. Symmetrically, Cond. 2 (rephrased as Cond. 2') requires that $\dot{\beta}$ is an lfp. Therefore, the IF/I coincidence yields a coincidence between an lfp and a gfp. This plays an important role in the next section.

▶ **Proposition 3.5.** *If p is a $\mathbf{CLat}_\wedge$-fibration then Cond. 2 and 3 in Def. 3.1 are equivalent to the following condition: there is a unique fixed-point $\mu\dot{F}$ of $(\beta^{-1})^*\dot{F} : \mathbb{E}_{\mu F} \to \mathbb{E}_{\mu F}$.*

IF/I Coincidence over Base IA-FC Coincidence. The IF/I coincidence (Def. 3.1) allows a simpler formulation in the special case where the IA-FC coincidence is *already there in the base category*. In this case, $\dot{\beta}^{-1}$ is final not only in a suitable fiber (Cond. 3 of Def. 3.1; cf. Prop. 3.3), but also *globally* in the total category $\mathbb{E}$. See [21, Appendix C.4] for details.

This special setting (the base IA-FC coincidence) is known to hold in domain-theoretic settings [13,27]. We use this setting (specifically the IA-FC coincidence in a Kleisli category [15]) in one of our applications (§7).

▶ **Theorem 3.6** (IF/I coincidence over the base coincidence). *Let $p : \mathbb{E} \to \mathbb{B}$ be a bifibration and $(p, F, \dot{F})$ be a tuple satisfying the IF/I coincidence. If there exists initial F-algebra $\beta : F(\mu F) \cong \mu F$ (in $\mathbb{B}$) such that β^{-1} is a final F-coalgebra, then there exists an initial $\dot{F}$-algebra $\dot{\beta} : \dot{F}(\mu\dot{F}) \cong \mu\dot{F}$ (in $\mathbb{E}$) above β such that $\dot{\beta}^{-1}$ is a final $\dot{F}$-coalgebra.*

4   IF/I Coincidence from Stable Chain Colimits

We now present our main observation, namely that the IF/I coincidence is a general phenomenon that relies only on a few mild assumptions. These assumptions include 1) that $\dot{F}$ is fibred (Def. 2.2) and 2) *stability* of chain colimits (Def. 4.1).

Here in §4, we restrict the underlying fibration $p \colon \mathbb{E} \to \mathbb{B}$ to a $\mathbf{CLat}_\wedge$-fibration over **Set** (Def. 2.3). This restriction simplifies proofs and technical developments. Nevertheless, we have a general coincidence theorem for not necessarily posetal fibrations; it is found in Appendix A. The general proof hinges on stable chain colimits, too.

The following is a key assumption. It is a fibrational adaptation of *pullback-stable colimit*, a notion studied in (higher) topos theory and categorical logic [9, 16, 22].

▶ **Definition 4.1** (stable chain colimits). We say that a fibration $p \colon \mathbb{E} \to \mathbb{B}$ has *stable chain colimits* if the following condition holds: for each $\lambda \in \mathrm{Ord}$ and each diagram $D \colon \mathrm{Ord}_{<\lambda} \to \mathbb{B}$,

1. $\mathbb{B}$ has a colimit of D. The i-th cocone component is denoted by $\kappa_i \colon Di \to \mathrm{colim}\, D$.
2. Moreover, for each object $P \in \mathbb{E}_{\mathrm{colim}\, D}$ above $\mathrm{colim}\, D$, we have $P \cong \mathrm{colim}\, \kappa_i^* P$, with the cartesian liftings $(\kappa_i^* P \to P)_{i \in \mathrm{Ord}_{<\lambda}}$ forming a colimiting cocone.

$$
\begin{array}{ll}
\mathbb{E} & \kappa_0^* P \longrightarrow \kappa_1^* P \longrightarrow \cdots \longrightarrow P \ (\cong \mathrm{colim}\, \kappa_i^* P) \\
{\scriptstyle \downarrow p} & \\
\mathbb{B} & D0 \longrightarrow D1 \longrightarrow \cdots \longrightarrow \mathrm{colim}\, D
\end{array}
$$

The first condition is equivalent to chain-cocompleteness. The situation of the second condition is illustrated as the above diagram. Stability requires that the upper cocone is colimiting. In the diagram, we note that morphisms $\kappa_i^* P \to \kappa_j^* P$ above $D(i \to j)$ are well-defined (where $i \leq j \leq \lambda$); they are induced by universality of the cartesian liftings $\kappa_j^* P \to P$.

Letting $\lambda = 0$ in Def. 4.1 yields the following property.

▶ **Lemma 4.2.** *Let $p \colon \mathbb{E} \to \mathbb{B}$ be a fibration with stable chain colimits. Then, all objects in $\mathbb{E}_0$ are initial in $\mathbb{E}$.*

▶ **Example 4.3.** The fibrations in Example 2.5 – $\mathbf{Pre} \to \mathbf{Set}$, $\mathbf{Pred} \to \mathbf{Set}$, $\mathbf{ERel} \to \mathbf{Set}$, and the domain fibration d^Ω for any complete lattice Ω – all have stable chain colimits. Non-examples are deferred to [21, Appendix B].

▶ **Theorem 4.4** (Main result). *Let $p \colon \mathbb{E} \to \mathbf{Set}$ be a $\mathbf{CLat}_\wedge$-fibration and $\dot{F}$ be a lifting of F along p. Assume further the following conditions:*
1. *there exists an initial F-algebra;*
2. *$\dot{F}$ is a fibred lifting of F;*
3. *p has stable chain colimits.*
Then $(p, F, \dot{F})$ satisfies the IF/I coincidence (Def. 3.1).

We prove the theorem in the rest of the section. Due to Prop. 3.5, it suffices to show that $(\beta^{-1})^* \dot{F} \colon \mathbb{E}_{\mu F} \to \mathbb{E}_{\mu F}$ has a unique fixed point, where β is an initial F-algebra. Cond. 1 in Thm. 4.4 yields that the initial chain of F converges and gives an initial F-algebra (Prop. 2.8 and 2.9).

We analyze the initial chains of F and $\dot{F}$, which is shown on the below.

$$
\begin{array}{ll}
\mathbb{E} & 0 \xrightarrow{\dot{\alpha}_{0,1}} \dot{F}0 \xrightarrow{\dot{\alpha}_{1,2}} \cdots \longrightarrow \dot{F}^\lambda 0 \xrightarrow{\dot{\alpha}_{\lambda,\lambda+1}} \dot{F}^{\lambda+1}0 \longrightarrow \cdots \\
{\scriptstyle p \downarrow} & \\
\mathbf{Set} & 0 \xrightarrow{\alpha_{0,1}} F0 \xrightarrow{\alpha_{1,2}} \cdots \longrightarrow F^\lambda 0 \xrightarrow{\alpha_{\lambda,\lambda+1}} \dot{F}^{\lambda+1}0 \longrightarrow \cdots
\end{array}
$$

Prop. 2.4 and Lem. 2.10 ensure that each chain morphism $\dot\alpha_{i,i+1}$ is above $\alpha_{i,i+1}$. Then, assuming that the initial chain of F converges in λ steps, the functor $(\beta^{-1})^*\dot{F}$ of our interest is equal to $\alpha_{\lambda,\lambda+1}^*\dot{F}$.

Fig. 1 is the key diagram about a unique fixed-point of $\alpha_{\lambda,\lambda+1}^*\dot{F}$. For simplicity, we write α for $\alpha_{\lambda,\lambda+1}$. We find the initial chain of $\dot{F}$ as its middle row; the initial chain of $\alpha^*\dot{F}$ as the bottom half of the last column; and the final chain of $\alpha^*\dot{F}$ as the top half. The other objects in the diagram are obtained by applying substitution to the last column.

Figure 1 IA-FC coincidence for $\mathbf{CLat}_\wedge$-fibrations, in Prop. 4.5. Here we write α for $\alpha_{\lambda,\lambda+1}$; the choice of λ is arbitrary (the initial $\dot{F}$-chain may not have stabilized).

The next result is the key technical observation. It says 1) the upper rows become closer to the initial $\dot{F}$-chain as we go below; and 2) symmetrically, the lower rows become closer to the same as we go up. Its proof is by transfinite induction; the stability assumption is crucially used in its limit case.

▶ **Proposition 4.5.** *Consider the setting of Thm. 4.4. Let λ be an arbitrary ordinal. We write $\alpha, \dot\alpha$ for $\alpha_{\lambda,\lambda+1}, \dot\alpha_{\lambda,\lambda+1}$ and $\top, \bot$ for the maximum and minimum of the complete lattice $\mathbb{E}_{F^\lambda 0}$. For each ordinal i, the objects $(\alpha^*\dot{F})^i\bot$ and $(\alpha^*\dot{F})^i\top$ above $F^\lambda 0$ are defined by the initial chain and the final chain of $\alpha^*\dot{F}$ (the last column of Fig. 1).*

Then we have $\alpha_{i,\lambda}^*(\alpha^*\dot{F})^i\bot = \dot{F}^i 0 = \alpha_{i,\lambda}^*(\alpha^*\dot{F})^i\top$ for each i with $i \leq \lambda$.

Proof sketch; a full proof is in [21, Appendix C.6]. The proof is by transfinite induction on i. The base case is clear because $\mathbb{E}_0$ includes only one object by Lemma 4.2 and the posetal assumption on p.

In the step case, fibredness of the lifting $\dot{F}$ lifts the equality for i, which is $\alpha_{i,\lambda}^*(\alpha^*\dot{F})^i\bot = \dot{F}^i 0 = \alpha_{i,\lambda}^*(\alpha^*\dot{F})^i\top$ (the induction hypothesis), to the desired equality for $i+1$.

The limit case is less obvious than the other cases. We rewrite the target objects (e.g. $\alpha_{i,\lambda}^*(\alpha^*\dot{F})^i\bot$) to chain colimits (e.g. $\mathrm{colim}_{j<i}\, \alpha_{j,\lambda}^*(\alpha^*\dot{F})^i\bot$) by stability of chain colimits, and use the fact that colimits of diagonal elements (e.g. $\mathrm{colim}_{j<i}\, \alpha_{j,\lambda}^*(\alpha^*\dot{F})^j\bot$) are equal to $\dot{F}^i 0$ by the induction hypothesis. See [21, Appendix C.6] for a full proof. ◀

Letting $i = \lambda$ in Prop. 4.5 yields that $(\alpha^*\dot{F})^\lambda\bot = \dot{F}^\lambda 0 = (\alpha^*\dot{F})^\lambda\top$. Therefore, both the initial and final chains of $\alpha^*\dot{F}$ (the last column in Fig. 1) converge in λ steps. We conclude that $\dot{F}^\lambda 0$ is both the lfp and gfp for $\alpha^*\dot{F}\colon \mathbb{E}_{F^\lambda 0} \to \mathbb{E}_{F^\lambda 0}$, hence is its unique fixed point.

Here are some consequences of the proposition. In the next result, note that the number of converging steps of F and that of $\dot{F}$ are not the same in general. See [21, Appendix B] for an example.

▶ **Corollary 4.6.** *Let* $p : \mathbb{E} \to \mathbf{Set}$ *be a* $\mathbf{CLat}_{\wedge}$*-fibration and* $\dot{F}$ *be a fibred lifting of* F *along* p. *Assume* p *has stable chain colimits. Then, the initial chain of* F *converges in* λ *steps if and only if that of* $\dot{F}$ *converges in* λ *steps.*

▶ **Corollary 4.7.** *In the setting of Cor. 4.6, if* F *has an initial algebra* α, *then any isomorphism* $\dot{F}P \to P$ *above* α *is an initial algebra of* $\dot{F}$.

We are finally in a position to prove our main theorem.

Proof of Thm. 4.4. Using Prop. 2.9 and Cor. 4.6, Cond. 1 ensures the existence of an ordinal λ such that both the initial chains of F and $\dot{F}$ converges in the steps. Then $\alpha_{\lambda,\lambda+1}^{-1}$ is an initial F-algebra by Prop. 2.8 and $\dot{F}^{\lambda}0$ is a unique fixed-point of $\alpha_{\lambda,\lambda+1}^{*}\dot{F}$ by Prop. 4.5. Prop. 3.5 concludes the proof.   ◀

5   Coincidence for Ω-Valued Predicates, Pure and Effectful

We instantiate the above categorical results to an important family of examples, namely Ω-*valued predicates* (Example 2.5). In this setting, a functor lifting $\dot{F}$ (§3) has a concrete presentation as an F-algebra, an observation that helps identification of many examples.

Besides the "pure" setting modeled by the fibration $\mathbf{Set}/\Omega \to \mathbf{Set}$, we also consider the "effectful" setting $\mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$, where effects are modeled by a monad $\mathcal{T}$ [23] with its lifting $\dot{\mathcal{T}}$ along d^{Ω}, and the base category is the Kleisli category for $\mathcal{T}$. The categorical construction of the fibration $\mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$ is described later in §5.2; the construction builds upon the recent results in [4].

The theoretical development here in §5 specializes that in §3–4, but it is still in abstract categorical terms. The theory in §5 paves the way to the concrete applications in §6–7.

5.1   Coincidence for Ω-Valued Predicates, the Pure Setting

We first focus on the domain fibration $d^{\Omega} : \mathbf{Set}/\Omega \to \mathbf{Set}$ (Example 2.5), where 1) a complete lattice Ω is regarded as a truth value domain, and 2) the fibration is regarded as that of Ω-valued predicates. If Ω is the two-element lattice $\mathbf{2} = \{\bot, \top\}$, then $d^{\mathbf{2}} : \mathbf{Set}/\mathbf{2} \to \mathbf{Set}$ is isomorphic to $\mathbf{Pred} \to \mathbf{Set}$.

Towards the IF/I coincidence for the fibration d^{Ω}, we first need to describe a fibred lifting $\dot{F}$ of F. It is induced by an F-algebra over Ω that is equipped with a suitable order structure.

▶ **Definition 5.1** (monotone algebra [4]). Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor and Ω be a complete lattice. We call $\sigma : F\Omega \to \Omega$ a *monotone F-algebra* over Ω if $i \leq_X i' \Rightarrow \sigma \circ Fi \leq_{FX} \sigma \circ Fi'$ holds for all $X \in \mathbf{Set}$ and all $i, i' \in \mathbf{Set}(X, \Omega)$.

▶ **Lemma 5.2** (from [4,7]). *Let* $F : \mathbf{Set} \to \mathbf{Set}$ *be a functor, and* Ω *be a complete lattice. There is a bijective correspondence between monotone F-algebras* $\sigma : F\Omega \to \Omega$ *and fibred liftings* $\dot{F}$ *of* F *along* d^{Ω}. *Specifically,* σ *gives rise to the lifting* $\dot{F}$ *given by* $\dot{F}(X \xrightarrow{x} \Omega) = (FX \xrightarrow{Fx} F\Omega \xrightarrow{\sigma} \Omega)$; *conversely,* $\dot{F}$ *gives rise to* $(F\Omega \xrightarrow{\sigma} \Omega) = \dot{F}(\Omega \xrightarrow{\mathrm{id}_{\Omega}} \Omega)$.

Application of §4 to a domain fibration is then easy.

▶ **Theorem 5.3** (coincidence for Ω-valued predicates, pure). *In the setting of Def. 5.1, let* $\sigma : F\Omega \to \Omega$ *be a monotone F-algebra. By Lem. 5.2,* σ *gives rise to a fibred lifting* $\dot{F}$ *of* F *along* d^{Ω}. *If there exists an initial F-algebra then* $(d^{\Omega}, F, \dot{F})$ *satisfies the IF/I coincidence.*

5.2 Coincidence for Ω-Valued Predicates, Effectful

In order to accommodate some concrete examples (those in §7 to be specific), we extend the above material to the setting with monadic effects.

We aim at the situation in (2), where the domain fibration d^Ω is Kleisli-embedded in the fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}} \colon \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$ on the right. The latter is the desired fibration of effectful computations and Ω-valued predicates; moreover, we extend a functor F and its lifting $\dot{F}$ for the Kleisli fibration, too.

$$\dot{F} \circlearrowright \mathbf{Set}/\Omega \xrightarrow{\dot{L}} \mathbf{Kl}(\dot{\mathcal{T}}) \circlearrowright \dot{F}_{\mathcal{T}}$$
$$\downarrow d^\Omega \qquad \downarrow d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}$$
$$F \circlearrowright \mathbf{Set} \xrightarrow{L} \mathbf{Kl}(\mathcal{T}) \circlearrowright F_{\mathcal{T}}$$
$$\tag{2}$$

The construction of the Kleisli fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}$ is via a *cartesian lifting* of the monad $\mathcal{T}$. It is defined to be a monad $(\dot{\mathcal{T}}, \dot{\eta}, \dot{\mu})$ on $\mathbf{Set}/\Omega$ such that 1) $\dot{\mathcal{T}}$ (as a functor) is a fibred lifting of the functor $\mathcal{T}$, and 2) $\dot{\eta}, \dot{\mu}$ are componentwise cartesian morphisms above η, μ, respectively. Then $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}} \colon \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$ is defined to be the evident extension of d^Ω to Kleisli categories, and is a fibration [4]. Cartesian liftings of $\mathcal{T}$ from $\mathbf{Set}$ to $\mathbf{Set}/\Omega$ bijectively correspond to Eilenberg-Moore (EM) $\mathcal{T}$-algebras, much like in Lem. 5.2.

▶ **Definition 5.4** (EM monotone algebra [4]). Let $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad and Ω be a complete lattice. A monotone $\mathcal{T}$-algebra $\tau : \mathcal{T}\Omega \to \Omega$ (where $\mathcal{T}$ is identified with its underlying functor) is called an *Eilenberg-Moore (EM) monotone $\mathcal{T}$-algebra* if $\mathrm{id}_\Omega = \tau \circ \eta_\Omega$ and $\tau \circ \mathcal{T}\tau = \tau \circ \mu_\Omega$. Here η and μ are the unit and multiplication of the monad $\mathcal{T}$.

▶ **Lemma 5.5** (from [4, Thm. 4.4]). *Let $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad and Ω be a complete lattice. There is a bijective correspondence between*
- *EM monotone $\mathcal{T}$-algebras τ, and*
- *Cartesian liftings $\dot{\mathcal{T}}$ of $\mathcal{T}$ that is itself a monad on $\mathbf{Set}/\Omega$.*

Specifically, τ gives rise to the lifting $\dot{\mathcal{T}}$ given by $\dot{\mathcal{T}}(X \xrightarrow{x} \Omega) = (\mathcal{T}X \xrightarrow{\mathcal{T}x} \mathcal{T}\Omega \xrightarrow{\tau} \Omega)$; conversely, $\dot{\mathcal{T}}$ gives rise to $(\mathcal{T}\Omega \xrightarrow{\tau} \Omega) = \dot{\mathcal{T}}(\Omega \xrightarrow{\mathrm{id}_\Omega} \Omega)$.

Let us now describe the fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}$ between Kleisli categories – it is the one on the right in (2). Recall that the *Kleisli category* $\mathbf{Kl}(\mathcal{T})$ of a monad $\mathcal{T}$ on $\mathbb{C}$ has the same objects as $\mathbb{C}$, and its morphisms from C to D are $\mathbb{C}$-morphisms $C \to \mathcal{T}D$ (often denoted by $C \nrightarrow D$). In view of Lem. 5.5, the Kleisli category $\mathbf{Kl}(\dot{\mathcal{T}})$ is described as follows:
- its objects are pairs $(X, f : X \to \Omega)$ where the latter is an Ω-valued predicate;
- its morphisms from $(X, f : X \to \Omega)$ to $(Y, g : Y \to \Omega)$ are $h : X \to \mathcal{T}Y$ such that $f \leq_X \dot{\mathcal{T}}g \circ h$ as shown below, where τ is the EM monotone $\mathcal{T}$-algebra that corresponds to the lifting $\dot{\mathcal{T}}$ (Lem. 5.5).

$$X \xrightarrow{h} \mathcal{T}Y$$
$$f \searrow \quad {}^{\leq_X} \quad \swarrow \dot{\mathcal{T}}g = \tau \circ \mathcal{T}g$$
$$\Omega$$

▶ **Lemma 5.6** (the fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}} \colon \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$ [4, Cor. 3.5]). *Let $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad, Ω be a complete lattice, and τ be an EM monotone $\mathcal{T}$-algebra. By Lem. 5.5, τ gives the fibred lifting $\dot{\mathcal{T}}$ of $\mathcal{T}$ such that $\dot{\mathcal{T}}$ is a monad.*
1. *The functor $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}} \colon \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$, defined as follows, is a posetal fibration: $(X \to \Omega) \longmapsto X$ on objects, and $(f : (X \to \Omega) \nrightarrow (Y \to \Omega)) \longmapsto f : X \nrightarrow Y$ on morphisms.*
2. *For each X in $\mathbf{Set}$, we have the isomorphism $(\mathbf{Set}/\Omega)_X \cong \mathbf{Kl}(\dot{\mathcal{T}})_{LX}$ between fibers. Here $L : \mathbf{Set} \to \mathbf{Kl}(\mathcal{T})$ is the Kleisli left adjoint that carries each object X to X.*

Now that we have described the fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}} : \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$, let us extend the functors $F, \dot{F}$ to $F_{\mathcal{T}}, \dot{F}_{\dot{\mathcal{T}}}$ (cf. (2)). We can do so by specifying how F and $\mathcal{T}$ interact.

▶ **Definition 5.7** (distributive law [24]). Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor and $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad with unit η and multiplication μ. A *distributive law* of F over $\mathcal{T}$ is a natural transformation $\lambda : F\mathcal{T} \Rightarrow \mathcal{T}F$ that makes the following diagrams commute.

$$
\begin{array}{ccc}
FX \xrightarrow{F\eta_X} F\mathcal{T}X & \quad F\mathcal{T}^2 X \xrightarrow{\lambda_{\mathcal{T}X}} \mathcal{T}F\mathcal{T}X \xrightarrow{\mathcal{T}\lambda_X} \mathcal{T}^2 FX \\
\eta_{FX} \searrow \quad \downarrow \lambda_X & \quad \downarrow F\mu_X \qquad\qquad\qquad \downarrow \mu_{FX} \\
\mathcal{T}FX & \quad F\mathcal{T}X \xrightarrow{\qquad\qquad \lambda_X \qquad\qquad} \mathcal{T}FX
\end{array}
$$

▶ **Lemma 5.8** (from [24]). *Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor, $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad and $L : \mathbf{Set} \to \mathbf{Kl}(\mathcal{T})$ be the left adjoint to the Kleisli category of $\mathcal{T}$. There is a bijective correspondence between distributive laws $\lambda : F\mathcal{T} \Rightarrow \mathcal{T}F$ and extensions $F_{\mathcal{T}} : \mathbf{Kl}(\mathcal{T}) \to \mathbf{Kl}(\mathcal{T})$ of F along L (that is, $F_{\mathcal{T}} \circ L = L \circ F$).*

The next lemma tells how to lift a distributive law λ of F over $\mathcal{T}$ to that of $\dot{F}$ over $\dot{\mathcal{T}}$. It follows from [4, Thm. 4.4].

▶ **Lemma 5.9.** *Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor, $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad, and Ω be a complete lattice. Consider a fibred lifting $\dot{F}$ of F corresponding to a monotone F-algebra $\sigma : F\Omega \to \Omega$ and a Cartesian lifting $\dot{\mathcal{T}}$ of $\mathcal{T}$ corresponding to an EM monotone $\mathcal{T}$-algebra $\tau : \mathcal{T}\Omega \to \Omega$ (see Lem. 5.2 and 5.5). Assume further that a distributive law $\lambda : F\mathcal{T} \Rightarrow \mathcal{T}F$ is compatible with σ and τ, in the sense that $\sigma \circ F\tau \leq \tau \circ \mathcal{T}\sigma \circ \lambda_\Omega$. Then this λ induces a distributive law $\dot{\lambda} : \dot{F}\dot{\mathcal{T}} \Rightarrow \dot{\mathcal{T}}\dot{F}$ of $\dot{F}$ over $\dot{\mathcal{T}}$ above λ.*

Finally, we obtain the fibrations and functors shown in (2).

▶ **Definition 5.10.** Assume the setting of Thm. 5.3. Let $\mathcal{T} : \mathbf{Set} \to \mathbf{Set}$ be a monad; τ be an EM monotone $\mathcal{T}$-algebra on Ω; and λ be a distributive law satisfying $\sigma \circ F\tau \leq \tau \circ \mathcal{T}\sigma \circ \lambda_\Omega$. We define $(d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}, F_{\mathcal{T}}, \dot{F}_{\dot{\mathcal{T}}})$ as follows.

- The EM monotone $\mathcal{T}$-algebra $\tau : \mathcal{T}\Omega \to \Omega$ gives rise to a Cartesian monad lifting $\dot{\mathcal{T}}$ of $\mathcal{T}$ along d^Ω (Lem. 5.5) and a fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}} : \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\mathcal{T})$ (Lem. 5.6).
- The distributive law $\lambda : F\mathcal{T} \Rightarrow \mathcal{T}F$ induces $F_{\mathcal{T}} : \mathbf{Kl}(\mathcal{T}) \to \mathbf{Kl}(\mathcal{T})$ such that $F_{\mathcal{T}}$ is an extension of F (in the sense of $F_{\mathcal{T}} \circ L = L \circ F$, Lem. 5.8).
- Because λ satisfies $\sigma \circ F\tau \leq \tau \circ \mathcal{T}\sigma \circ \lambda_\Omega$, Lem. 5.9 canonically induces a distributive law $\dot{\lambda} : \dot{F}\dot{\mathcal{T}} \Rightarrow \dot{\mathcal{T}}\dot{F}$.
- This distributive law $\dot{\lambda}$ induces an extension $\dot{F}_{\dot{\mathcal{T}}} : \mathbf{Kl}(\dot{\mathcal{T}}) \to \mathbf{Kl}(\dot{\mathcal{T}})$ of $\dot{F}$ (Lem. 5.8), which is also a lifting of $\dot{F}$ along $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}$.
- (Optional) If λ satisfies the equality $\sigma \circ F\tau = \tau \circ \mathcal{T}\sigma \circ \lambda_\Omega$ (instead of the inequality $\leq$ required in the above), then $\dot{F}_{\dot{\mathcal{T}}}$ is a fibred lifting of $F_{\mathcal{T}}$.

The above technical material (mainly from [4]) allows us to state this section's main result.

▶ **Theorem 5.11** (coincidence for Ω-valued predicates, effectful). *In the setting of Def. 5.10, if there exists an initial F-algebra then $(d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}, F_{\mathcal{T}}, \dot{F}_{\dot{\mathcal{T}}})$ satisfies the IF/I coincidence.*

The proof of Thm. 5.11 is *not* a straightforward application of the general results in §4 to the fibration $d^\Omega_{\mathcal{T},\dot{\mathcal{T}}}$. Notice, for example, that fibredness of the lifting $\dot{F}_{\dot{\mathcal{T}}}$ is not mandatory in Def. 5.10, while it is required in the general IF/I coincidence result (Thm. 4.4). Indeed, the lifting $\dot{F}_{\dot{\mathcal{T}}}$ is not fibred in our application in §7, so Thm. 4.4 does not apply to it.

Our proof of Thm. 5.11 instead goes via the "pure" fibration d^{Ω} (on the left in (2)): using the fact that the left adjoint L preserves initial chains, we essentially lift the IF/I coincidence from pure (d^{Ω}) to effectful ($d^{\Omega}_{\mathcal{T},\hat{\mathcal{T}}}$). We count this proof (in [21, Appendix C.10]) as one of our main contributions.

6   Application 1: Probabilistic Liveness by Submartingales

We use the IF/I coincidence results in §3–5 to derive a new proof method for probabilistic liveness – more concretely, we derive the method as an instance of Thm. 5.3. Liveness properties are usually witnessed by *ranking supermartingales*; see e.g. [10, 29]. Restricting to finite trees, we show that probabilistic liveness can also be witnessed by an invariant-like *sub*martingale (as opposed to *super*martingale) notion.

Here is the class of probabilistic systems that we analyze. It is restricted for the simplicity of presentation; accommodating more expressive formalisms is easy by changing a functor.

▶ **Definition 6.1** (finite probabilistic binary tree). A *finite probabilistic binary tree* is a finite binary tree such that

- each internal node n is labeled with either ✓ or ?; and
- each edge is labeled with a real number $p \in [0,1]$, in such a way that two outgoing edge-labels sum to 1. See (3).

$$
\begin{array}{ccc}
& n & \\
{}^{p}\swarrow & & \searrow^{1-p} \\
n_1 & & n_2
\end{array}
\tag{3}
$$

We restrict to *finite* trees; here is one application scenario that justifies it. We think of those probabilistic trees as models of systems with internal coin toss. We assume that there is some timeout mechanism that forces the termination of those systems, that is, that termination of the target system is guaranteed by some external means. Such mechanism forcing finiteness is common in real-world systems.

The liveness property we are interested in is eventually reaching a state labeled with ✓. More precisely, we are interested in the probability of eventually seeing ✓. The following invariant-like witness notion gives a guaranteed lower bound for the probability in question. It is derived from the IF/I coincidence; unlike ranking supermartingales, it does not use natural numbers or ordinals.

▶ **Definition 6.2** (IF/I submartingale). Let t be a finite probabilistic binary tree; the set of its nodes is denoted by N. We say $f : N \to [0,1]$ is an *IF/I submartingale* if it satisfies the following.

1. $f(n) = 0$ for each leaf node n.
2. For each internal node n labeled with ?, let its children and their edge labels be as shown in (3). Then we have

$$f(n) \le p \cdot f(n_1) + (1-p) \cdot f(n_2).$$

The direction of the inequality is indeed that of a *sub*martingale: the current value is a lower bound of the expected next value. Note that there is no condition for $f(n)$ if n is an internal node labeled with ✓. In this case, $f(n)$ can be set to 1 to improve the lower bound.

▶ **Theorem 6.3.** *In the setting of Def. 6.2, assume f is an IF/I submartingale. Then, identifying the tree t with the Markov chain with suitable probabilistic branching, the probability of eventually reaching a node labeled with ✓ from the root is at least $f(r_t)$ where r_t is the root node of t.*

The proof of Thm. 6.3 is in [21, Appendix C.11]. The main step is to apply the following to Thm. 5.3 in order to obtain a categorical data $(d^{[0,1]}, F^{\mathrm{ptr}}, \dot{F}^{\mathrm{ptr}})$ satisfying the IF/I coincidence:

- a complete lattice Ω is $[0,1]$ with the usual order between real numbers;
- a set functor F is $F^{\mathrm{ptr}} = \mathbf{1} + \{\checkmark, ?\} \times [0,1] \times (-)^2$; and
- a monotone F^{ptr}-algebra $\sigma : F^{\Sigma}[0,1] \to [0,1]$ is σ^{ptr} defined as follows:

$$
\sigma^{\mathrm{ptr}}(x) = \begin{cases} 0 & \text{if } x = * \in \mathbf{1} \\ 1 & \text{if } x = (\checkmark, p, a, b) \\ pa + (1-p)b & \text{if } x = (?, p, a, b). \end{cases}
$$

7   Application 2: Witnesses for Bottom-Up Tree Automata

We present an application of the IF/I coincidence in §3 to tree automata, using the results in §5 as an interface. In this paper we restrict to *bottom-up* tree automata, although a similar theory can be developed for top-down ones.

We restrict the ranked alphabet Σ used for trees to $\Sigma = \Sigma_0 \cup \Sigma_2$, where operations in Σ_0 are 0-ary and those in Σ_2 are binary. This restriction is for simplicity and not essential.

▶ **Definition 7.1** ((finite) Σ-trees). A Σ-*tree* t is a tuple (N, r_t, c_t) where N is a set of nodes, $r_t \in N$ is a root node and $c_t : N \to \Sigma_0 + \Sigma_2 \times N \times N$ is a function which determines labels and next nodes: if $c_t(n) = s \in \Sigma_0$ then n is a leaf node labeled with $s \in \Sigma_0$, and if $c_t(n) = (s, n_1, n_2)$ then n is an internal node labeled with $s \in \Sigma_2$ and the next nodes of n are n_1 and n_2. A *finite* Σ-*tree* is a Σ-tree which has only finitely many nodes.

▶ **Definition 7.2** (bottom-up tree automaton). A *bottom-up tree automaton* is a quadruple $\mathcal{A} = (\Sigma_0 \cup \Sigma_2, Q, \delta, q_{\mathsf{F}})$, where 1) $\Sigma_0 \cup \Sigma_2$ is a ranked alphabet; 2) Q is a set of states; 3) $\delta : \Sigma_0 + \Sigma_2 \times Q \times Q \to \mathcal{P}Q$ is a transition function (note the nondeterminism modeled by the powerset $\mathcal{P}Q$); and 4) $q_{\mathsf{F}} \in Q$ is an accepting state.

A *run* of $\mathcal{A}$ over a Σ-tree t is a function ρ from nodes n of t to states $\rho(n) \in Q$ such that 1) $\rho(n) \in \delta(s)$ for each leaf node n with $c_t(n) = s$, and 2) $\rho(n) \in \delta\big(s, \rho(n_1), \rho(n_2)\big)$ for each internal node n with $c_t(n) = (s, n_1, n_2)$.

A finite Σ-tree t is *accepted* by $\mathcal{A}$ if there is a run ρ of $\mathcal{A}$ over t such that $\rho(r_t) = q_{\mathsf{F}}$.

Note that allowing multiple accepting states does not change the theory because of the nondeterminism in transition functions.

Upside-Down Witness for Acceptance.   For an acceptance of a single Σ-tree by a bottom-up tree automaton, the IF/I coincidence in §3 and §5.1 (the pure setting) yields the following (invariant-like, top-down) witness notion.

▶ **Definition 7.3.** Let $\mathcal{A} = (\Sigma_0 \cup \Sigma_2, Q, \delta, q_{\mathsf{F}})$ be a bottom-up tree automaton, and $t = (N, r_t, c_t)$ be a finite Σ-tree. We say $f : N \to \mathcal{P}Q$ is an *acceptance invariant* if
1. for each leaf node n with $c_t(n) = s$, we have $f(n) \subseteq \delta(s)$;
2. for each internal node n with $c_t(n) = (s, n_1, n_2)$, we have $f(n) \subseteq \bigcup_{q_1 \in f(n_1), q_2 \in f(n_2)} \delta\big(s, q_1, q_2\big)$;
3. for the root node r_t of t, we have $q_{\mathsf{F}} \in f(r_t)$.

An acceptance invariant assigns a *predicate* $f(n)$ to each node n, and the constraints on f runs in the top-down manner. The proof of Thm. 7.4 is in [21, Appendix C.12], where we identify suitable categorical constructs (a fibration and functors) and apply the results in §5.1.

▶ **Theorem 7.4** (acceptance witness for a finite tree)**.** *In the setting of Def. 7.3, if there exists an acceptance invariant $f : N \to \mathcal{P}Q$, then $\mathcal{A}$ accepts the finite Σ-tree t.*

Upside-Down Witness for Model Checking.   We extend the above theory from acceptance (of a single tree) to *model checking* (whether every tree generated by a system is accepted). Besides its practical relevance, the model checking problem is categorically interesting. Specifically, for the results here, we use the extended categorical framework in §5.2 (IF/I coincidence in presence of *effects*) and Thm. 3.6 (coincidence lifting).

▶ **Definition 7.5** (generative tree automaton $\mathcal{C}$, its language $L_{\mathcal{C}}^{\text{fin}}$, and model checking)**.** A *generative tree automaton* is $\mathcal{C} = (\Sigma_0 \cup \Sigma_2, X, c, x_0)$, where 1) $\Sigma_0 \cup \Sigma_2$ is a ranked alphabet; 2) X is a set of states; 3) $c \colon X \to \mathcal{P}(\Sigma_0 + \Sigma_2 \times X \times X)$ is a transition function (note the powerset operator $\mathcal{P}$); and 4) $x_0 \in X$ is an initial state.

Let $t = (N, r_t, c_t)$ be a (possibly infinite) Σ-tree. A *run* of $\mathcal{C}$ over t is a function $\rho \colon N \to X$, assigning a state to each node, such that 1) $\rho(r_t) = x_0$ for the root node r_t; 2) $s \in c(\rho(n))$ for each leaf node n with $c_t(n) = s$; and 3) $\big(s, \rho(n_1), \rho(n_2)\big) \in c(\rho(n))$ for each internal node n with $c_t(n) = (s, n_1, n_2)$.

We say that a Σ-tree t is *generated* by $\mathcal{C}$ if there is a run ρ of $\mathcal{C}$ over t. The set of all Σ-trees generated by $\mathcal{C}$ is denoted by $L_{\mathcal{C}}$; the set of all *finite* Σ-trees generated by $\mathcal{C}$ is $L_{\mathcal{C}}^{\text{fin}}$.

The *model checking problem* takes a generative tree automaton $\mathcal{C}$ and a bottom-up tree automaton $\mathcal{A}$ (Def. 7.2) as input, and asks if every finite Σ-tree in $L_{\mathcal{C}}^{\text{fin}}$ is accepted by $\mathcal{A}$.

Note that we restrict to finite trees here. One possible justification is an external mechanism that forces termination, much like in §6.

Our general theory of the IF/I coincidence derives the following (invariant-like, top-down) witness notion for model checking (where the specification is a bottom-up tree automaton).

▶ **Definition 7.6** (model checking invariant)**.** Let $\mathcal{A} = (\Sigma_0 \cup \Sigma_2, Q, \delta, q_{\mathsf{F}})$ be a bottom-up tree automaton, and let $\mathcal{C} = (\Sigma_0 \cup \Sigma_2, X, c, x_0)$ be a generative tree automaton. We say $f : X \to \mathcal{P}Q$ is a *model checking invariant* if it satisfies the following.
1. $f(x) \subseteq \bigcap_{a \in c(x)} \delta_f(a)$ for each $x \in X$. Here $\delta_f : \Sigma_0 + \Sigma_2 \times X \times X \to \mathcal{P}Q$ is defined by 1) $\delta_f(s) = \delta(s)$ for $s \in \Sigma_0$; 2) $\delta_f(s, x_1, x_2) = \bigcup_{q_1 \in f(x_1), q_2 \in f(x_2)} \delta(s, q_1, q_2)$ for $s \in \Sigma_2$.
2. $q_{\mathsf{F}} \in f(x_0)$.

▶ **Theorem 7.7.** *In the setting of Def. 7.6, assume that there exists a model checking invariant $f : X \to \mathcal{P}Q$. Then, $\mathcal{A}$ accepts every finite Σ-tree $t \in L_{\mathcal{C}}^{\text{fin}}$ generated by $\mathcal{C}$.*

The proof is in [21, Appendix C.13]. The nondeterminism on the system side ($\mathcal{C}$ in Def. 7.5) requires to work in the effectful setting (§5.2). Another challenge is that the relevant functor lifting is not fibred (cf. the last item in Def. 5.10); we use the coincidence lifting (Thm. 3.6) to deal with it, where the required base coincidence comes from coalgebraic trace semantics [15].

8   Conclusions and Future Work

We presented our IF/I coincidence, which is a general categorical framework for the coincidence of initial algebras and final coalgebras, a classic topic in computer science. The IF/I coincidence is formulated in fibrational terms, and this occurs in the fiber over an initial algebra; it is therefore understood as the coincidence of logical lfp and gfp specifications. Relying on mild assumptions of fibred liftings and stable chain colimits, the IF/I coincidence accommodates many examples. As applications, we derived seemingly new verification methods for probabilistic liveness and tree automata.

The proofs in §6–7 suggest the possibility of a structural theory of the IA-FC coincidence, where unique fixed points are pulled back along coalgebra homomorphisms. We will pursue this structural theory, together with its practical consequences.

Another direction of future work is to formalize the relationship between the current fibrational approach to the IA-FC coincidence, and the homset enrichment approach in [5, 11, 12, 15, 27].

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A   IF/I Coincidence for general fibrations

Here we show the following theorem similar to Thm. 4.4 for general fibrations.

▶ **Theorem A.1** (IF/I coincidence for general fibrations). *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration; assume that $\mathbb{E}$ and $\mathbb{B}$ are chain-cocomplete. Let $\dot{F}$ be a lifting of F along p. Assume further the following conditions:*

1. *The initial chain of $\dot{F}$ converges.*
2. *$\dot{F}$ is a fibred lifting of F.*
3. *p has stable chain colimits.*
4. *p strictly preserves chain colimits.*
5. *Substitution in p preserves chain colimits in fibers.*

Then $(p, F, \dot{F})$ satisfies the IF/I coincidence.

The theorem follows from the next technical observation.

▶ **Proposition A.2.** *In the setting of Thm. A.1, assume further that the initial chain of $\dot{F}$ converges in λ steps. Consider the initial chains:*

$$
\begin{array}{cccccccc}
\mathbb{E} & & 0 \xrightarrow{\dot{\alpha}_{0,1}} \dot{F}0 \xrightarrow{\dot{\alpha}_{1,2}} \cdots \longrightarrow \dot{F}^\lambda 0 \underset{\cong}{\xrightarrow{\dot{\alpha}_{\lambda,\lambda+1}}} \dot{F}^{\lambda+1}0 \longrightarrow \cdots \\
p \downarrow & & \\
\mathbb{B} & & 0 \xrightarrow{\alpha_{0,1}} F0 \xrightarrow{\alpha_{1,2}} \cdots \longrightarrow F^\lambda 0 \underset{\cong}{\xrightarrow{\alpha_{\lambda,\lambda+1}}} \dot{F}^{\lambda+1}0 \longrightarrow \cdots
\end{array}
$$

where 0 in the two lines denote initial objects in $\mathbb{E}$ and in $\mathbb{B}$, respectively. Note that both $\dot{\alpha}_{\lambda,\lambda+1}$ and $\alpha_{\lambda,\lambda+1}$ are isomorphisms: the former is by the convergence assumption; the latter is by the assumption and Lem. 2.10. Therefore their inverses are initial algebras by Prop. 2.8.

In this setting, $\dot{\alpha}_{\lambda,\lambda+1}$ is a final object of $\mathbf{Coalg}(\dot{F})_{\alpha_{\lambda,\lambda+1}}$ (cf. Def. 3.1).

From now on, we aim to prove Prop. A.2.

To show finality of $\dot{\alpha}_{\lambda,\lambda+1}$ in $\mathbf{Coalg}(\dot{F})_{\alpha_{\lambda,\lambda+1}}$, we first claim the existence of a morphism from an arbitrary $\dot{F}$-coalgebra $\gamma : P \to \dot{F}P$ in $\mathbf{Coalg}(\dot{F})_{\alpha_{\lambda,\lambda+1}}$ to $\dot{\alpha}_{\lambda,\lambda+1}$. The next lemma shows a construction of such a morphism p_λ by transfinite induction along initial chains.

$$
\begin{array}{ccc}
\mathbb{E} & & \dot{F}^\lambda 0 \xrightarrow{\dot{\alpha}_{\lambda,\lambda+1}} \dot{F}^{\lambda+1}0 \\
\Big\vert & & \quad p_\lambda \uparrow \qquad \qquad \uparrow \dot{F}p_\lambda \\
\Big\vert \, p & & \quad P \xrightarrow{\quad \gamma \quad} \dot{F}P \\
\Big\downarrow & & \\
\mathbb{B} & & F^\lambda 0 \xrightarrow{\alpha_{\lambda,\lambda+1}} F^{\lambda+1}0
\end{array}
$$

This construction exploits the singleton property of $\mathbb{E}_0$ (Lem. 4.2) in the base case, fibredness of the lifting $\dot{F}$ in the step case, and stability of chain colimits in the limit case.

▶ **Lemma A.3.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration with stable chain colimits. Assume that $\mathbb{E}$ and $\mathbb{B}$ are chain-cocomplete and p strictly preserves chain colimits. Let $\dot{F}$ be a fibred lifting of F along p.*

For each ordinal λ and each coalgebra $\gamma : P \to \dot{F}P$ above $\alpha_{\lambda,\lambda+1}$ (or equivalently, $\gamma \in \mathbf{Coalg}(\dot{F})_{\alpha_{\lambda,\lambda+1}}$), there exists a morphism $p_\lambda : \gamma \to \dot{\alpha}_{\lambda,\lambda+1}$ in $\mathbf{Coalg}(\dot{F})_{\alpha_{\lambda,\lambda+1}}$.

Proof. Let $(k_{j,i} : \alpha^*_{j,\lambda}P \to \alpha^*_{i,\lambda}P)_{j \le i \le \lambda}$ denote those morphisms induced from $(\alpha^*_{i,\lambda}P \to P)_{i \le \lambda}$ via their universality as cartesian liftings. We construct vertical morphisms $(p_i : \alpha^*_{i,\lambda}P \to \dot{F}^i 0)_{i \le \lambda}$ such that $p_i \circ k_{j,i} = \dot{\alpha}_{j,i} \circ p_j$ for all i, j with $j \le i \le \lambda$. Such a $(p_i)_{i \le \lambda}$ makes the diagram below commute. (The rightmost square commutes, see [21, Appendix C.14] for details.)

$$
\begin{array}{ccccccccc}
\mathbb{E} & & 0 & \xrightarrow{\dot{\alpha}_{0,1}} & \dot{F}0 & \xrightarrow{\dot{\alpha}_{1,2}} & \cdots \longrightarrow & \dot{F}^{\lambda}0 & \xrightarrow{\dot{\alpha}_{\lambda,\lambda+1}} & \dot{F}^{\lambda+1}0 \\
\Big\downarrow & & {\scriptstyle p_0}\Big\uparrow & & {\scriptstyle p_1}\Big\uparrow & & & {\scriptstyle p_\lambda}\Big\uparrow & & {\scriptstyle \dot{F}p_\lambda}\Big\uparrow \\
p & & \alpha_{0,\lambda}^* P & \xrightarrow{k_{0,1}} & \alpha_{1,\lambda}^* P & \xrightarrow{k_{1,2}} & \cdots \longrightarrow & \alpha_{\lambda,\lambda}^* P = P & \xrightarrow{\gamma} & \dot{F}P \\
\Big\downarrow & & & & & & & & & \\
\mathbb{B} & & 0 & \xrightarrow{\alpha_{0,1}} & F0 & \xrightarrow{\alpha_{1,2}} & \cdots \longrightarrow & F^{\lambda}0 & \xrightarrow{\alpha_{\lambda,\lambda+1}} & F^{\lambda+1}0
\end{array}
$$

Then this p_λ is what we want. The construction of $(p_i)_{i\le\lambda}$ is by the following transfinite induction on i.

- (Base case) Lem. 4.2 says $\alpha_{0,\lambda}^* P \cong 0$ in $\mathbb{E}$ and this isomorphism is vertical because both $p\alpha_{0,\lambda}^* P$ and $p0$ are 0 in $\mathbb{B}$. We define p_0 as this isomorphism.
- (Step case) If i is a successor ordinal, we define p_i by

$$
\alpha_{i,\lambda}^* P \xrightarrow{(\star)} (F\alpha_{i-1,\lambda})^* \dot{F}P \xrightarrow[\cong]{\xi} \dot{F}\alpha_{i-1,\lambda}^* P \xrightarrow{\dot{F}p_{i-1}} \dot{F}^i 0
$$

  where ξ is from fibredness of $\dot{F}$ and $(\star)$ is induced as follows by universality of a cartesian lifting.

$$
\begin{array}{ccc}
\alpha_{i,\lambda}^* P & \xrightarrow{\quad\overline{\alpha_{i,\lambda}}\quad} & P \searrow{\scriptstyle\gamma} \\
{\scriptstyle(\star)}\Big\downarrow & & \searrow \\
(F\alpha_{i-1,\lambda})^* \dot{F}P & \xrightarrow{\quad\overline{F\alpha_{i-1,\lambda}}\quad} & \dot{F}P
\end{array}
$$

$$
F^i 0 \xrightarrow{\alpha_{i,\lambda}} F^\lambda 0 \xrightarrow{\alpha_{\lambda,\lambda+1}} F^{\lambda+1}0.
$$

  Note that $F\alpha_{i-1,\lambda} = \alpha_{\lambda,\lambda+1} \circ \alpha_{i,\lambda}$ by the definition of α. We can prove $\dot{\alpha}_{j,i} \circ p_j = p_i \circ k_{j,i}$ for all $j \le i$ by transfinite induction on j. See [21, Appendix C.14].
- (Limit case) If i is a limit ordinal, we define p_i by the stability of chain colimits. By applying chain colimit stability of p to $\alpha_{i,\lambda}^* P$ above $F^i 0 = \mathrm{colim}_{j<i} F^j 0$ (by Def. 2.7, see below), we have $\alpha_{i,\lambda}^* P \cong \mathrm{colim}_{j<i}\, \alpha_{j,i}^*(\alpha_{i,\lambda}^* P) \cong \mathrm{colim}_{j<i}\, \alpha_{j,\lambda}^* P$. For all l,j with $l < j < i$, by the induction hypothesis, we have $\dot{\alpha}_{l,i} \circ p_l = \dot{\alpha}_{j,i} \circ \dot{\alpha}_{l,j} \circ p_l = \dot{\alpha}_{j,i} \circ p_j \circ k_{l,j}$. Hence $(\dot{F}^i 0, (\alpha_{j,i} \circ p_j)_{j<i})$ is a cocone over $(j \mapsto \alpha_{j,\lambda}^* P)$, as shown below. We define p_i as the mediating morphism from a colimit, as in the following diagram.

$$
\begin{array}{c}
\cdots \longrightarrow \dot{F}^l 0 \longrightarrow \dot{F}^j 0 \longrightarrow \cdots \quad\nearrow \dot{F}^i 0 \text{ (colim.)} \\
{\scriptstyle p_l}\uparrow \qquad {\scriptstyle p_j}\uparrow \qquad\qquad {\scriptstyle p_i} \\
\cdots \longrightarrow \alpha_{l,\lambda}^* P \longrightarrow \alpha_{j,\lambda}^* P \longrightarrow \cdots \quad\searrow \alpha_{i,\lambda}^* P \text{ (colim.)} \\
\cdots \longrightarrow F^l 0 \xrightarrow{\alpha_{l,j}} F^j 0 \xrightarrow{\alpha_{j,i}} F^i 0 \longrightarrow \cdots
\end{array}
$$

This concludes the proof.   ◀

For Prop. A.2, it remains to show the uniqueness of $p_\lambda : \gamma \to \dot{\alpha}_{\lambda,\lambda+1}$. The uniqueness does not immediately follow from the construction of p_λ in Lem. A.3.

Our uniqueness proof (in the proof of Prop. A.2 shown later), we work on a suitable chain in the fiber $\mathbb{E}_{F^\lambda 0}$, defined as follows.

The following fact (cf. [19, Prop. 9.2.2 and Exercise 9.2.4]) shows $\mathbf{CLat}_\wedge$-fibrations have properties suitable for colimits.

▶ **Proposition A.4.** *Let $p : \mathbb{E} \to \mathbb{B}$ be an opfibration. Assume the base category $\mathbb{B}$ has colimits of shape $\mathbb{I}$. Then the following statements are equivalent.*

1. *Each fiber of the opfibration p has colimits of shape $\mathbb{I}$.*
2. *The total category $\mathbb{E}$ has colimits of shape $\mathbb{I}$ and p strictly preserves them.*

▶ **Notation A.5** ($\dot{\alpha}, \alpha$). In the setting of Lem. A.3, let us fix λ to be a converging ordinal of the initial $\dot{F}$-chain, in the sense that $\dot{\alpha}_{\lambda,\lambda+1} : \dot{F}^{\lambda}0 \xRightarrow{\cong} \dot{F}^{\lambda+1}0$ is an isomorphism. In the rest of the section, we write $\dot{\alpha}, \alpha$ for $\dot{\alpha}_{\lambda,\lambda+1}, \alpha_{\lambda,\lambda+1}$, respectively. Then $\alpha : F^{\lambda}0 \xRightarrow{\cong} F^{\lambda+1}0$ is an isomorphism, too.

▶ **Definition A.6.** In Prop. A.2, we define a chain

$$P \xrightarrow{\beta_{0,1}} \alpha^* \dot{F} P \xrightarrow{\beta_{1,2}} (\alpha^* \dot{F})^2 P \xrightarrow{\beta_{2,3}} \cdots$$

by repeated application of $\alpha^* \dot{F}$. The whole chain resides in the fiber $\mathbb{E}_{F^{\lambda}0}$, as shown in (4).

$$
\begin{array}{cc}
\vdots & \vdots \\
\uparrow & \uparrow \\
(\alpha^* \dot{F})^2 P & \dot{F}(\alpha^* \dot{F})^2 P \\
{\scriptstyle \beta_{1,2}}\uparrow \quad \xrightarrow{\bar{\alpha}} & \uparrow{\scriptstyle \dot{F}\beta_{1,2}} \\
\alpha^* \dot{F} P & \dot{F}\alpha^* \dot{F} P \\
{\scriptstyle \beta_{0,1}}\uparrow \quad \xrightarrow{\bar{\alpha}} & \uparrow{\scriptstyle \dot{F}\beta_{0,1}} \\
P \xrightarrow{\quad \gamma \quad} & \dot{F} P \\
\\
F^{\lambda}0 \xrightarrow{\quad \alpha \quad} F^{\lambda+1}0 &
\end{array}
\tag{4}
$$

The precise definition is as follows. It is similar to Def. 2.7, but starting from P (instead of from 0) calls for some care.

- (Objects) $(\alpha^* \dot{F})^i P$ is given for each $i \in \mathrm{Ord}$: $(\alpha^* \dot{F})^0 P = P$, $(\alpha^* \dot{F})^{i+1} P = \alpha^* \dot{F}((\alpha^* \dot{F})^i P)$, and $(\alpha^* \dot{F})^i 0 = \mathrm{colim}_{j<i}(\alpha^* \dot{F})^j 0$ for a limit ordinal i.
- (Morphisms) The morphism $\beta_{i,i+1} : (\alpha^* \dot{F})^i 0 \to (\alpha^* \dot{F})^{i+1} 0$ for each ordinal i is defined as follows.
  - (Base case) $\beta_{0,1} : P \to \alpha^* \dot{F} P$ is induced from $\gamma : P \to \dot{F} P$ by universality of the cartesian lifting $\bar{\alpha} : \alpha^* \dot{F} P \to \dot{F} P$. See (4).
  - (Step case) $\beta_{i+1,i+2}$ is defined by $\alpha^* \dot{F}\beta_{i,i+1}$.
  - (Limit case) $\beta_{i,i+1} : (\alpha^* \dot{F})^i P \to (\alpha^* \dot{F})^{i+1} P$ for a limit ordinal i is induced by universality of $(\alpha^* \dot{F})^i P = \mathrm{colim}_{j<i}(\alpha^* \dot{F})^j P$. Prop. A.4 ensures this colimit vertex is above $F^{\lambda}0$.

  We have defined $\beta_{j,j+1}$ for each ordinal j. This induces morphisms $\beta_{i,j} : (\alpha^* \dot{F})^i P \to (\alpha^* \dot{F})^j P$ for each $i < j$ in a straight-forward manner: one repeats the step and limit cases; when j is a limit ordinal, $\beta_{i,j}$ is the cocone component to $(\alpha^* \dot{F})^j P = \mathrm{colim}_{k<j}(\alpha^* \dot{F})^k P$.

▶ **Lemma A.7.** *In the setting of Prop. A.2, let us assume that λ is a converging ordinal in the initial chain of $\dot{F}$, and adopt Notation A.5.*
Fig. 2 shows the following constructs.

- *The morphism $\gamma : P \to \dot{F} P$ above α induces the chain $P \xrightarrow{\beta_{0,1}} \alpha^* \dot{F} P \xrightarrow{\beta_{1,2}} (\alpha^* \dot{F})^2 P \to \cdots$ as in Def. A.6.*
- *The last chain induces, for each ordinal l such that $l < \lambda$, the chain*

$$\alpha_{l,\lambda}^* P \xrightarrow{\alpha_{l,\lambda}^* \beta_{0,1}} \alpha_{l,\lambda}^* \alpha^* \dot{F} P \xrightarrow{\alpha_{l,\lambda}^* \beta_{1,2}} \alpha_{l,\lambda}^* (\alpha^* \dot{F})^2 P \to \cdots$$

*above $F^l 0$, via the substitution along $\alpha_{l,\lambda}$.*

- For each ordinal i, we obtain a $\dot{F}$-coalgebra as follows. It is above α; it is denoted by γ_i.

$$
\begin{array}{ccc}
(\alpha^*\dot{F})^i P & \xrightarrow{\gamma_i := \overline{\alpha} \circ \beta_{i,i+1}} & \dot{F}(\alpha^*\dot{F})^i P \\[2mm]
F^\lambda 0 & \xrightarrow[\cong]{\alpha} & F^{\lambda+1} 0
\end{array}
\tag{5}
$$

- We apply Lem. A.3 to the last coalgebras γ_i, using each of them in place of the coalgebra γ in Lem. A.3. Following the proof of Lem. A.3, we obtain vertical morphisms $(p_j^i : \alpha_{j,\lambda}^*(\alpha^*\dot{F})^i P \to \dot{F}^j 0)_{j \leq \lambda}$ for each ordinal i.

In this case, for each i such that $i \leq \lambda$, (i) p_i^i is an isomorphism; (ii) $p_i^l = p_i^m \circ \alpha_{i,\lambda}^* \beta_{l,m}$ for all l, m with $l \leq m$.

See [21, Appendix C.15] for the proof.
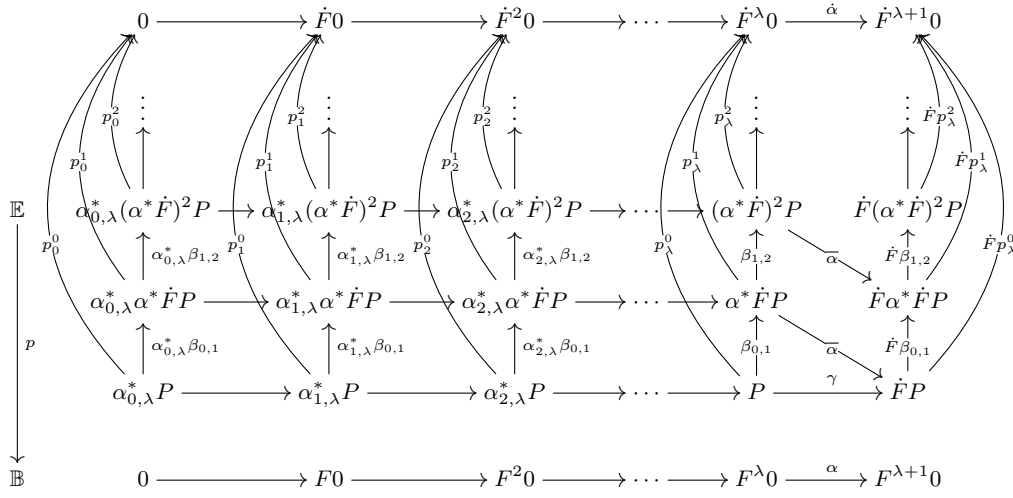
Figure 2 A diagram for Lem. A.7.

The last lemma shows that the $\dot{F}$-coalgebra $\gamma_i : (\alpha^*\dot{F})^i P \to \dot{F}(\alpha^*\dot{F})^i P$ gets closer to $\dot{\alpha}$ as i gets larger, with a particular consequence that γ_λ is isomorphic to $\dot{\alpha}$ (via p_λ^λ). This is used in the following proof of Prop. A.2.

Proof of Prop. A.2. Let γ be an arbitrary coalgebra $P \to \dot{F}P$ above $\alpha = \alpha_{\lambda,\lambda+1}$ (Notation A.5). Lem. A.3 shows the existence of a vertical morphism from γ to $\dot{\alpha}_{\lambda,\lambda+1}$. We only need to show the uniqueness of morphisms. Let f be an arbitrary vertical morphism from γ to $\dot{\alpha}_{\lambda,\lambda+1}$. The isomorphic correspondence in Lem. 3.4 carries $f : (P \xrightarrow{\gamma} \dot{F}P) \to (\dot{F}^\lambda 0 \xrightarrow{\dot{\alpha}} \dot{F}^{\lambda+1} 0)$ in $\mathbf{Coalg}(\dot{F})_\alpha$ to $f : (P \xrightarrow{\beta_{0,1}} \alpha^*\dot{F}P) \to (\dot{F}^\lambda 0 \xrightarrow{\delta} \alpha^*\dot{F}^{\lambda+1} 0)$ in $\mathbf{Coalg}(\alpha^*\dot{F})$, where δ is the mediating morphism from $\dot{\alpha}$ by universality of the cartesian lifting $\overline{\alpha} : \alpha^*\dot{F}^{\lambda+1} 0 \to \dot{F}^{\lambda+1} 0$.

Using the above f in $\mathbf{Coalg}(\alpha^*\dot{F})$, we consider the following two chains and a morphism between them. Everything here is above $F^\lambda 0$; cf. (4). δ is an isomorphism since the initial chain of $\dot{F}$ converges in λ steps.

$$
\begin{array}{ccccccccc}
\dot{F}^\lambda 0 & \xrightarrow[\delta]{\cong} & (\alpha^*\dot{F})\dot{F}^\lambda 0 & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & (\alpha^*\dot{F})^\lambda \dot{F}^\lambda 0 & \xrightarrow{\cong} & (\alpha^*\dot{F})^{\lambda+1} \dot{F}^\lambda 0 \\[1mm]
f \uparrow & & \uparrow (\alpha^*\dot{F})f & & & & (\alpha^*\dot{F})^\lambda f \uparrow & & (\alpha^*\dot{F})^{\lambda+1} f \uparrow \\[1mm]
P & \xrightarrow{\beta_{0,1}} & (\alpha^*\dot{F})P & \xrightarrow{} & \cdots & \xrightarrow{\beta_{\lambda-1,\lambda}} & (\alpha^*\dot{F})^\lambda P & \xrightarrow{\beta_{\lambda,\lambda+1}} & (\alpha^*\dot{F})^{\lambda+1} P
\end{array}
\tag{6}
$$

It follows easily that $\beta_{\lambda,\lambda+1}$ is the inverse of an initial $\alpha^*\dot{F}$-algebra. This is essentially because 1) γ_λ is isomorphic to $\dot{\alpha}$ (see the paragraph that follows Lem. A.7); and 2) (the inverse of) $\gamma_\lambda = \overline{\alpha} \circ \beta_{\lambda,\lambda+1}$ corresponds to (the inverse of) $\beta_{\lambda,\lambda+1}$ in the isomorphic correspondence $\mathbf{Alg}(\dot{F})_{\alpha^{-1}} \cong \mathbf{Alg}(\alpha^*\dot{F})$ in Lem. 3.4.

Consider the rightmost square in (6). By universality of the initial $\alpha^*\dot{F}$-algebra $(\beta_{\lambda,\lambda+1})^{-1}$, $(\alpha^*\dot{F})^\lambda f$ is unique; therefore the composite $(\alpha^*\dot{F})^\lambda f \circ \beta_{\lambda-1,\lambda} \circ \cdots \circ \beta_{0,1}$ (on the left in (6)) is uniquely determined. By the commutativity of (6) and the fact that all the morphisms in the first row are isomorphisms, this uniquely determines f, too. ◀